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AN ATAVISTIC LIE ALGEBRA

The infinite-dimensional Lie algebra

$$\boxed{[J^a_{m_1,m_2}, J^b_{n_1,n_2}] =}$$

$$e^{is(m_1e^{-a}n_2 - m_2e^a n_1)} J^{a+b}_{m_1+e^a n_1, m_2+e^{-a}n_2} - e^{is(n_1e^{-b}m_2 - n_2e^b m_1)} J^{a+b}_{n_1+e^b m_1, n_2+e^{-b}m_2},$$

contains as subalgebras or limits **most Lie algebras utilized in physics**: $GL(N)$, Classical Lie, Moyal, Poisson, Virasoro, Vertex... (Arbitrary indices, a, b, $m_1, m_2, ..$ and parameter s, unless restricted by further expediency).

• Underlain by the noncompact oscillator group $\mathcal{G}$ (or $\mathcal{H}_4$, W Miller: the solvable, rank 2, dimension 4, Lie group generated by the oscillator creation and annihilation operators, their Heisenberg commutator—central—and the occupation number operator).

• It satisfies the **Jacobi identity**; evident, as it merely amounts to the antisymmetrization of the associative (**finite dimensional Lie group**) **product**,

$$J^a_{m_1,m_2} J^b_{n_1,n_2} = e^{is(m_1 e^{-a} n_2 - m_2 e^a n_1)} J^{a+b}_{m_1+e^a n_1,\ m_2+e^{-a} n_2}.$$

Associativity means $\quad (J^a_{m_1,m_2} J^b_{n_1,n_2}) J^c_{k_1,k_2} = J^a_{m_1,m_2} (J^b_{n_1,n_2} J^c_{k_1,k_2}).$

• The symmetrization of this product into an anticommutator further yields a consistent **graded extension** of the infinite Lie algebra. (The generators of the infinite-dim Lie algebras are exponentials of finite-dim Lie algebras.) Center: $J^0_{0,0} = J^a_{m_1,m_2} J^{-a}_{-e^{-a}m_1,-e^a m_2}$.

• Applications in deconstruction, noncommutative QFT, and possibly twisted CFT, and...

REALIZATIONS

On the unit T^2, (x, p range from 0 to 1), Groenewold's 1946 **associative** $\star$-product,

$$\star \equiv e^{-is(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)} ,$$

allows strings of operators of the form $f(x,p)\star$, for any functions $f(x,p)$, to be equivalently evaluated, indifferently to the grouping of the non-commutative multiplications chosen.

In a Fourier mode basis, for integers m_1, m_2,

$$M_{m_1,m_2} \equiv \exp(i(m_1 x + m_2 p)) \star ,$$

the standard product law follows,

$$M_{m_1,m_2} M_{n_1,n_2} = \exp\left(is(m_1 n_2 - m_2 n_1)\right) \; M_{m_1+n_1,m_2+n_2} ,$$

yielding the **Sine Algebra** when antisymmetrized. (The finite Lie group underlying this product is just the dimension 3 Heisenberg group.)

• Consider a phase-space-area-preserving dilation operator $D(a)$, which braids associatively as

$$D(a)\ f(x,p) = f(e^a x, e^{-a} p)\ D(a)\ ,$$

$$D(a)D(b) = D(a+b), \qquad\qquad D(0) = \mathbb{1}.$$

$\rightsquigarrow$ It formally commutes with the star product,

$$D(a)\ \star = \star\ D(a) = \exp\left(-is(\overleftarrow{\partial}_x \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_x)\right)\ \exp\left(a(x\ \overrightarrow{\partial}_x - p\ \overrightarrow{\partial}_p)\right),$$

This one acts differently on the left and right arguments (branes), virtually like a sheared star product.

$\rightsquigarrow$ The Atavistic Algebra elements $J^a_{m_1,m_2}$ may be constructed out of $M_{m_1,m_2} D(a)$,

$$J^a_{m_1,m_2} = e^{i(m_1 x + m_2 p)}\ e^{s(m_1\partial_p - m_2\partial_x)}\ e^{a(x\partial_x - p\partial_p)}.$$

$\rightsquigarrow$ sequential rescalings and shifts of a function's variables, and overall multiplication by a phase,

$$J^a_{m_1,m_2} f(x,p) = e^{i(m_1 x + m_2 p)}\ f(e^a(x - sm_2),\ e^{-a}(p + sm_1)).$$

But, since x and p commute,

$\rightsquigarrow$ **Coherent state realization**,

$$\boxed{J^{a}_{m_1,m_2} = e^{im_1\alpha^\dagger - 2sm_2\alpha}\ e^{a\ \alpha^\dagger\alpha}}\ ,$$

underlain by (dim 4, rank 2) oscillator group.

- $a \to 0$ yields the (FFZ) sine algebra, (Moyal Bracket algebra), hence classical Lie algebras, Poisson, Virasoro, ...

$m_2 \to 0$, yields the (FZ) vertex algebra.

UNFAITHFUL REPRESENTATIONS

No faithful matrix (finite-dim) representations. For restricted parameters, unfaithful ones are based on Sylvester's (1882) clock and shift matrices for $GL(p)$.

Choose $s = -\pi/p$ for an odd prime p, $\looparrowright$ $\exp(-2is) \equiv \omega = e^{2\pi i/p}$, with $\omega^p = 1$; take integer subscripts mod p, $m_j = 0, 1, 2, ..., p-1$; and rescaled superscripts to be integer mod $p-1$, $\tilde{a} \equiv a/\ln 2 = 0, 1, 2, ..., p-2$, recalling cyclicity: $2^{p-1} = 1 \bmod p$, for any odd prime integer p.

$\looparrowright$ The product now reads

$$J^{\tilde{a}}_{m_1,m_2} J^{\tilde{b}}_{n_1,n_2} = \omega^{(2^{\tilde{a}} m_2 n_1 - 2^{p-1-\tilde{a}} m_1 n_2)/2} \; J^{\tilde{a}+\tilde{b}}_{m_1 + 2^{\tilde{a}} n_1, \; m_2 + 2^{p-1-\tilde{a}} n_2} .$$

Represented by Sylvester's $p \times p$ unitary unimodular matrices,

$$Q_{rt} = \omega^r \delta_{r,t} \; , \qquad\qquad P_{rt} = \delta_{r+1,t} \; ,$$

for r, t defined mod p, $r = 0, 1, 2, ..., p-1$. $\rightsquigarrow$

$$Q^p = P^p = \mathbb{1}, \qquad\qquad PQ = \omega \; QP.$$

The complete set of p^2 unitary unimodular $p \times p$ matrices

$$M_{(m_1,m_2)} \equiv \omega^{m_1 m_2/2} Q^{m_1} P^{m_2}, \quad \Longrightarrow \quad [M_{(m_1,m_2)}]_{rt} = \omega^{m_1(r+m_2/2)} \delta_{r+m_2,t} ,$$

where $M^{\dagger}_{(m_1,m_2)} = M_{(-m_1,-m_2)}$, and $\mathrm{Tr} M_{(m_1,m_2)} = p\, \delta_{m_1,0}\, \delta_{m_2,0}$, suffice to span the **group** of $GL(p)$.

But since

$$M_{(m_1,m_2)} M_{(n_1,n_2)} = \omega^{(m_2 n_1 - m_1 n_2)/2}\ M_{(m_1+n_1,m_2+n_2)} ,$$

they further satisfy the **Lie algebra** of $SU(p)$, a restriction of the Sine Algebra (FFZ, 1989),

$$\boxed{[M_{(m_1,m_2)}, M_{(n_1,n_2)}] = 2i\ \sin\left(\frac{\pi}{p}(m_2 n_1 - m_1 n_2)\right) M_{(m_1+n_1,m_2+n_2)}} .$$

In addition, consider the discrete scaling (doubling) matrix (Vourdas)

$$R_{rt} \equiv \delta_{2r,t} , \qquad R^{p-1} = \mathbb{1}, \qquad R^{\dagger} = R^T = R^{p-2},$$

for r, t defined mod p: $\quad r, t = 0, 1, 2, ..., p-1$.

The cyclic structure holds by virtue of the identity $2^{p-1} = 1$ mod p.

$$RQR^{p-2} = Q^2, \qquad R^{p-2} PR = P^2.$$

$$R^{\tilde{a}} Q^{m_1} P^{m_2} R^{p-1-\tilde{a}} = Q^{2^{\tilde{a}} m_1} P^{2^{p-1-\tilde{a}} m_2},$$

$\looparrowright$ p-dimensional unitary matrix representation,

$$\boxed{\mathcal{J}^{\tilde{a}}_{m_1,m_2} \equiv M_{(m_1,m_2)} R^{\tilde{a}} = \omega^{m_1 m_2/2} Q^{m_1} P^{m_2} R^{\tilde{a}}.}$$

However, since Sylvester's basis is complete, R is representable in terms of the above p^2 Ms—in fact, it is the phased sum of all $p \times p$ matrices M, normalized by p, since, $\forall m_1, m_2$,

$$\mathrm{Tr}\ M_{(m_1,m_2)} R = \omega^{-3 m_1 m_2/2}\ . \qquad \Longrightarrow$$

$$\boxed{p \mathcal{J}^{1}_{0,0} - \sum_{m_1,m_2} \omega^{-3 m_1 m_2/2}\ \mathcal{J}^{0}_{m_1,m_2} = 0}\ ,$$

is represented trivially: the representations displayed are **not faithful**.

$\divideontimes$ Generalization of $D(a)$ to **linear canonical transformations** $Sp(2)$ (Bogoliubov transformations)—associativity preserved.

• Application to **systems where there is transvection shear between the two sides** of a $\star$-product, e.g. on a deconstruction brane lattice link, squeezed states, Bogoliubov transformations...